Philadelphia University

Lecture Notes for 650364

Probability & Random Variables

Chapter 1:

Lecture 4: Combined Experiments and Bernoulli Trials

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Probability

1) Introduction
2) Set Definitions
3) Set Operations
4) Probability Introduced Through Sets and Relative Frequency
5) Joint and Conditional Probability
6) Total Probability and Bayes' Theorem
7) Independent Events
8) **Combined Experiments**
9) **Bernoulli Trials**

8)Combined Experiments

- ✓ A **combined experiment** consists of forming a single experiment by suitably combining individual experiments called **subexperiments**.
- ✓ **Combined Sample Space**:
  - Consider two subexperiments with sample spaces **S1** and **S2**.
  - We form a **new sample space** called the **combined sample space** whose elements are all the ordered pairs **(s1, s2).**
  - The combined sample space is denoted

$$S = S1 \times S2$$

- ✓ For a sequence of **n** events in which the first event can occur in **k1** ways and the second event can occur in **k2** ways and the third event can occur in **k3** ways, and so on, the total number of ways the sequence can occur is $k1 \cdot k2 \cdot k3 \ldots \cdot kn$
- ✓ **Example:** In an experiment of flipping a coin and rolling a die, the sample spaces of subexperiments are given by:

$$S1 = \{H, T\} \text{ and } S1 = \{1, 2, 3, 4, 5, 6\}$$

The combined sample space $S = S1 \times S2$ becomes

S = {(H,1), (H,2), (H,3), (H,4), (H,5), (H,6), (T,1), (T,2), (T,3), (T,4), (T,5), (T,6)}

For example, the probability of event **A= {T, 5}** equals **1/12**

- ✓ **Example:** In an experiment of flipping a coin twice, the sample spaces of subexperiments are given by:

$S1 = \{H, T\}$ and $S2 = \{H,T\}$

The combined sample space **$S = S1 \times S2$** becomes

$S = \{(H,H), (H,T), (T,H), (T,T)\}$

Fundamental Counting Techniques (Combinatorial analysis)

1) Permutations:

- ✓ In experiments often involve **multiple trials** in which outcomes are elements of a finite sample space and they are **not replaced** after each trial the number of possible sequences of the outcomes is often important.
- ✓ An **arrangement** of **n** distinct objects in a specific order is called a permutation.
- ✓ The number of permutations of **n** objects using all the objects is **n!**
- ✓ The **number of sequences**, or **permutations**, of **r** elements taken from **n** elements when **order of occurrence is important** is given by:

$$P_r^n = n(n-1)(n-2)\dots(n-r+1)$$
$$= \frac{n!}{(n-r)!}, \qquad r = 1, 2, \dots, n$$
$$\text{where} \qquad n! = n\cdot(n-1)\cdot(n-2)\cdots 1$$

- ✓ **Example:** How many permutations are there for four cards taken from a 52-card deck?

$$P_4^{52} = \frac{52!}{(52-4)!} = (52)(51)(50)(49) = 6{,}497{,}400$$

- ✓ **Example:** In how many different ways can **6 people** be arranged in a row for a photograph?
    - ○ **Solution:** This is a permutation of 6 objects.

      Hence $6! = 6 * 5 * 4 * 3 * 2 * 1 = 720$ ways.
- ✓ **Example:** In how many different ways can **3 people** be arranged in a row for a photograph if they are selected from a group of **5 people**?
    - ○ **Solution:** Since 3 people are being selected from 5 people and arranged in a **specific order**, $n = 5, r = 3$. Hence, there are

$$ {}_5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5 \cdot 4 \cdot 3 \cdot \cancel{2!}}{\cancel{2!}} = 5 \cdot 4 \cdot 3 = 60 \text{ ways} $$

- ✓ **Example**: How many three-digit code can be made where all digits are unique? The possible digits are the numbers **0** through **9**.

$$ P_3^{10} = \frac{10!}{(10-3)!} = (10)(9)(8) = 720 \quad \text{Codes} $$

2) Combinations:

- ✓ The **number of sequences, or combinations**, of **r** elements taken from **n** elements when **order of occurrence is not important** (for example, **ABC=ACB=BAC**) is given by:

$$ \binom{n}{r} = \frac{n!}{(n-r)!\,r!} $$

The numbers $\binom{n}{r}$ are called binomial coefficients

- ✓ Suppose **two** letters are selected from the four letters, **A, B, C**, and **D.**

  The different **permutations** are shown on the left and the different **combinations** are shown on the right.

PERMUTATIONS				COMBINATIONS	
AB	BA	CA	DA	AB	BC
AC	BC	CB	DB	AC	BD
AD	BD	CD	DC	AD	CD

- ✓ **Example:** How many sequences are there for four cards taken from a 52-card deck (if the order of cards is not important)?

$$\binom{52}{4} = \frac{52!}{(52-4)!\,4!} = \frac{(52)(51)(50)(49)}{(4)(3)(2)(1)} = 270{,}725$$

- ✓ **Example**: In a classroom, there are **8 women** and **5 men**. A committee of **3 women and 2 men** is to be formed for a project. How many different possibilities are there?
  - **Solution:** In this case, you must select **3 women from 8 women** and **2 men from 5 men**. Since the word **"and"** is used, **multiply the answers**.

$$
\begin{aligned}
{}_8C_3 \cdot {}_5C_2 &= \frac{8!}{(8-3)!3!} \cdot \frac{5!}{(5-2)!2!} \\
&= \frac{8!}{5! \cdot 3!} \cdot \frac{5!}{3! \cdot 2!} \\
&= \frac{8 \cdot 7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!} \cdot 3 \cdot 2 \cdot 1} \cdot \frac{5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!} \cdot 2 \cdot 1} = 56 \cdot 10 \\
&= 560
\end{aligned}
$$

✓ **Example:** How many **three-digit code** can be made where the order of digits is not important? The possible digits are the numbers **0** through **9**.

$$\binom{10}{3} = \frac{10!}{(10-3)!3!} = \frac{(10)(9)(8)}{(3)(2)(1)} = 120 \quad \text{Codes}$$

9)Bernoulli Trials

✓ In the theory of probability and statistics, a **Bernoulli trial** is a random experiment with exactly **two possible outcomes**, **success or failure**, flipping a coin, hitting or missing a target, passing or failing an exam, in which the probability of success is the same every time the experiment is conducted.

- ✓ For this type of experiment, we let **A** be the elementary event having one of the two possible outcomes with probability **p** and its **complement** $\bar{A}$ with probability **1-p**.

$$P(A) = p \; , \; P(\bar{A}) = 1 - p$$

- ✓ If the basic experiment is repeated N times , such repeated experiments are called **Bernoulli trials**.
- ✓ The **probability** that event **A** occurs exactly **k** times out of the **N** trails is given By

$$P\{A \text{ occurs exactly } k \text{ times}\} = \binom{N}{k} p^k (1-p)^{N-k}$$

- ✓ **Example:** A submarine attempts to sink an aircraft carrier. It will be successful only if two or more torpedoes hit the carrier. If the sub fires three torpedoes and the probability of a hit is 0.4 for each torpedo, what is the probability that the carrier will be sunk?

$$P\{\text{exactly no hits}\} = \binom{3}{0}(0.4)^0(1-0.4)^3 = 0.216$$

$$P\{\text{exactly one hit}\} = \binom{3}{1}(0.4)^1(1-0.4)^2 = 0.432$$

$$P\{\text{exactly 2 hits}\} = \binom{3}{2}(0.4)^2(1-0.4)^1 = 0.288$$

$$P\{\text{exactly 3 hits}\} = \binom{3}{3}(0.4)^3(1-0.4)^0 = 0.064$$

The answer we desire is

$$\begin{aligned} P\{\text{carrier sunk}\} &= P\{\text{two or more hits}\} \\ &= P\{\text{exactly 2 hits}\} + P\{\text{exactly 3 hits}\} = 0.352 \end{aligned}$$

- ✓ **Example:** student is known to arrive late for class **30%** of the time, if the class meets five times a week. Find: the probability that the student is late for **at least** four classes in a given week; the probability that the student will not be late at all during a given week.

$$P\{\text{student is late for at least 4 classes}\} =$$
$$= P\{\text{exactly 4 classes}\} + P\{\text{exactly 5 classes}\}$$
$$= \binom{5}{4}(0.3)^4(1-0.3)^1 + \binom{5}{5}(0.3)^5(1-0.3)^0$$

✓ Calculation rules of probability - summary:

Sum rule

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= P(A) + P(B) \quad \text{(if } A \text{ and } B \text{ are mutually exclusive)} \end{aligned}$$

Product rule

$$\begin{aligned} P(A \cap B) &= P(A)\, P(B|A) \\ &= P(A)\, P(B) \quad \text{(if } A \text{ and } B \text{ are independent)} \end{aligned}$$

Total probability

$$P(A) = \sum_i P(B_i)\, P(A|B_i) \quad \text{(if } B_i\text{'s form a decomposition)}$$

Bayes' formula

$$P(B|A) = \frac{P(A|B)\, P(B)}{P(A)}$$

Extended Bayes' formula

$$P(B_i|A) = \frac{P(A|B_i)\, P(B_i)}{\sum_j P(A|B_j)\, P(B_j)} \quad \text{(if } B_i\text{'s form a decomposition)}$$